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**Discussion Papers in
Economics and Econometrics**

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Conclave

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Abstract

A committee is choosing from two alternatives. If required supermajority is not reached, voting is repeated indefinitely, although there is a cost of delay. Under suitable assumptions the equilibrium analysis provides a sharp prediction. The result can be interpreted as a generalization of the seminal median voter theorem known from the simple majority case. If supermajority is required instead, then the power to select the outcome moves from the median voter to the more extreme voters. Normative analysis indicates that the simple majority is strictly inferior to some supermajority. Even if unanimity is a bad voting rule, voting rules close to unanimity may be efficient. The more likely it is to have a very many almost indifferent voters and some very opinionated ones, the more stringent supermajority is required for efficiency.

Keywords: supermajority, qualified majority, repeated voting, conclave, War of Attrition.

JEL Classification Codes: D63, D72, D74

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1 Introduction

Consider a decision-making conclave – a committee locked in a room choosing from two alternatives, either of which may replace the status quo, through a supermajority of, say, 2/3. If voters do not reach a decision in the first round of voting, then they go to the next round, and so on, until the sufficient majority is reached. However, since they are locked in a room, delay or deadlock is increasingly costly to every player. How does the outcome of this process change if the required supermajority is adjusted to, say, 3/4? Which rule offers a better chance of a correct choice?

Apart from providing an equilibrium prediction in such mechanisms, the main contribution of this paper is to identify reasons why both simple majority and unanimity may be subefficient. In the simple majority case, the median voter is pivotal and – as it represents some "typicality" of preferences – it is often understood that her choice would implement an efficient alternative. A formal version of this argument was made by Rae (1969) under the assumption that voters' preferences have the same intensity. In contrast to that, this study allows preferences to have varying intensities. In equilibrium, the greater supermajority is required, the more extreme voters become pivotal. In other words, the outcome depends on a combination of these more extreme preferences, which is a measure of preference typicality different than the median. The main result is this: the combination of this measure correlates better with the mean preference, than the median preference does. Since the mean preference represents efficiency, one is interested in maximizing this correlation.

It has also been argued that one of the main reasons why unanimous consent is not a good voting method is because it may lead to delay or deadlock, for instance, through obstinacy.[1] This study shows that unanimity may be a bad voting rule even if a decision is reached without any delay. The reason is a different incarnation of the same fact as above: the combination of the most extreme preferences correlates very poorly with the average preference.

These results, along with the main assumptions of the model, will be discussed further after a few examples of this voting institution.

Examples. Some of the most important collective decisions emerge through such a process, either de jure or de facto.

Often, selecting a candidate by a committee resembles a conclave. It can

[1] Black (1963), p. 147, Buchanan and Tullock (1962), ch. 8., Barry (1965) p. 242-9.

be argued that choosing the President of the European Council, selection of the CEO of a corporation, agreeing on job candidates in hiring committees, all require repeated attempts to form a sufficient consensus. Vacancy or interregnum is often the worst outcome, particularly if the committee members are not allowed to leave the room for as long as the vacancy is not filled, and the only remaining question is which candidate should be selected.

Papal conclave – election of the Pope, the leader of the Roman Catholic Church – seems to be a fairly close example of this type of voting. A candidate requires 2/3 majority in the College of Cardinals, and the voting is repeated indefinitely until such support is reached. The Cardinals are locked "with the key" (*cum clave*) for the duration of the process; they receive food and can sleep, but otherwise they are forced to a rather monotonous lifestyle, which, long-term, must be increasingly unbearable. Despite the secrecy involved in the process, the voting rules are well-known and their evolution quite well-documented (Baumgartner (2003)). The requirement of 2/3 majority was written down as a rule by the Third Lateran Council of 1179. Since then, the only serious challenge came in 1996, when John Paul's new election constitution allowed the electors to switch to simple majority after 30-33 unsuccessful ballots (Baumgartner (2003)), which effectively sanctioned the simple majority as the rule. The John Paul's successor, Benedict XVI, changed the rules back to 2/3 majority in 2007. This study will provide an argument why a supermajority is likely to be superior to the simple majority.

Another example of this voting method is trial by jury. The jury has to arrive to a decision to acquit or to convict, and since unanimity or almost unanimity is required in most jurisdictions, this is obtained through repeated voting, intertwined with often lengthy deliberations among the jury members. Like in a papal conclave, the jury members are locked for the duration of the process.

The analysis below can shed some light on some types of negotiations as well. In open-ended negotiations, the involved parties have enough time to propose new alternatives and bargain, while the status quo is not necessarily the worst outcome. The model in this paper does not fit this scenario. However, when negotiations requiring a supermajority are summoned to address an emergency or a crisis, then it is likely that they instantaneously reach a stage in which a binary agenda is established, everyone agrees that waiting is costly, and status quo is the least preferable outcome. Many international organizations use various supermajority rules (e.g. the Treaty of Lisbon in the

European Union has replaced unanimity by less stringent qualified majority rules), and therefore this situation may occur from time to time.

Summary of the results. There are positive and normative questions asked in this paper.

Positive analysis, contained in Section 4, shows that preference intensities of the voters are the key determinants of the voting outcome. To be more precise, suppose that two alternatives are A and B, suppose that n is the size of the committee and suppose that supermajority $n+1-m$ is required, where the key parameter m is the minimal blocking minority. That is, m voters voting in unison are able to prevent an alternative from being selected; fewer than m is not enough. Voters have different preference intensities, so they can be sorted from the one who prefers A the most, to the one who prefers B the most. Take two *pivotal* voters, the mth voter from the A side and the mth voter from the B side and check which one prefers her alternative more – that alternative will be selected in equilibrium without any delay.

This result can be expressed in terms of one variable. If the mth voter from the A side prefers A more than the mth voter from the B side prefers B (thus leading to the selection of A), then the average of the values of these two pivotal voters will "prefer" A. This average is called the mth quasi-midrange because it is the midrange point of a sample whose $m-1$ most extreme elements from each side were truncated. The equilibrium characterization could be called the quasi-midrange voter theorem, the special case of which is the median voter theorem.

Section 5 turns to a normative analysis of what the best minimal blocking minority m is, if the voting system has to be decided before the voters learn their preferences. This is investigated for the case of frequent voting and asymptotically large committees and for symmetric parent distributions of voters' preferences. Unanimity is as bad as flipping a coin, simple majority is better, but there is an intermediate supermajority that is even better. The key statistical result behind this last claim is that a sample mean does not correlate with a sample median as well as with some other intermediate quasi-midranges. Since the mean represents constrained efficiency,[2] and the quasi-midrange represents equilibrium of an intermediate supermajority system, relying on the median is subefficient, as compared to the optimal intermediate quasi-midrange. As far as the unanimity system is concerned, the midrange

[2] The term "constrained" reflects the restriction that only supermajority systems are considered.

(without "quasi") correlates with the mean particularly poorly, and hence relying on it is particularly hopeless.

What supermajority is constrained efficient depends on a parent distribution of voters preferences, and, specifically, on the relationship between tail/extreme preferences versus central/indifferent preferences. Simple majority is the best system if preference intensities are similar enough. Even if unanimity is the worst voting method, systems close to unanimity may be optimal if (i) there is a high overall probability that voters are indifferent, but (ii) very extreme preferences can occur. This may sound somewhat paradoxical, but the intuition is straightforward. If such is the parent distribution, then most of the voters are indifferent. Preferences of a few very strong-minded voters determine what the efficient decision is, and so these voters should determine the outcome. Large supermajority requirement gives them that power.

As any theoretical model, this one can also be criticized on the basis of some modelling choices and lack of realism. Section 6 attempts to address some of these criticisms. Among other issues, it compares the asymptotic outcomes with simulated outcomes for relatively small committees. Examples considered suggest that the results obtained for the asymptotic cases are remarkably close to the ones obtained for committees consisting of one or two dozens members. Clearly, it is nonsensical to organize a conclave with millions of voters, but the asymptotic analysis presented in this paper offers a simple and convenient benchmark.

Summary of the main assumptions. Firstly, players' preferences are commonly known. Thus, this paper does not study information aggregation. Largely thanks to this assumption, there is no delay in equilibrium. This assumption is less restrictive that it may seem at first. Often, an essential element of repeated voting procedures is a possibility to exchange opinions and to present evidence. The organization of jury voting is intentionally conducive to deliberations. In papal conclave, voting rounds alternate with sermons by senior Cardinals and periods of reflection and dialogue. Even if voters enter the process with some private information, they will reveal it in equilibrium, provided that there exists a technology to present credible evidence and that there is ample time to do that (see also comments on Ponsati and Sakovics (1996) below).

Secondly, it is assumed that as the game progresses, the cost of waiting becomes large enough to dominate static preferences for one of the alternatives. Specifically, after a long enough wait, a voter will prefer to sacrifice the prospect of getting her favorite alternative next round by accepting the less preferred alternative this round. This assumption seems to be justified, given the format of the voting process. One should expect that, as time passes, voters locked in a room would become increasingly desperate to reach a decision. For example, members of a recruitment committee may argue vigorously for a few hours, but will become more cooperative when lunch is delayed. Even if Cardinals in a papal conclave, or jury members in jury trials, etc., receive food and time to sleep, after a few days or weeks they too may lose their patience to continue the process. In crisis negotiations, the cost of waiting is increasing fast, almost by definition.

2 Existing literature

The merits of unanimity versus simple majority rule in collective decision making is a classical topic. Following Buchanan's and Tullock's (1962) two-stage approach to constitution design, Rae (1969) provides a normative analysis of voting systems. He argues that simple majority is the best voting rule because it selects an alternative that a representative voter is most likely to prefer after she learns her preferences. This result heavily relies on the assumption that voters have preferences of equal intensity (see also May (1952)). The present study can be seen as an attempt to provide a parallel result when voters have preferences of varying intensity.

Studies of supermajority (such as Black (1948), Aghion at al. (2004), Holden (2009)), often envision the following 'static' scenario: there is some status-quo, which can be changed by some large enough supermajority.[3] In this class of models, Caplin and Nalebuff (1991) show that the proposal most preferred by the mean voter is unbeatable under approximately 63%-majority rule. The model below studies supermajority rules in a very different context: every voter agrees that status quo is inferior and voting is repeated until this status quo is replaced by one of the alternatives.

In the literature on dynamic multiplayer decision-making, the closest paper to this study is Ponsati and Sakovics (1996). They study a similar economic environment with many players, two alternatives and costly waiting,

[3]Messner and Polborn (2004) consider an overlapping-generations model in which a supermajority rule needed for future reforms is selected at the constitutional design phase. Older generations, who would pay immediate costs but would not reap future benefits from reforms, are in favour of selecting a supermajority rule that preserves status quo.

but they focus exclusively on equilibrium characterization. They assume a "simple timing game" in which a player can continue supporting her favorite alternative or yield once and for all. The second key assumption in their model is that a player cannot observe how many players supporting the other alternative have yielded. In contrast to this, in the model below voters can switch their votes as they please and they observe the full history of voting. Since Ponsati and Sakovics (1996) assume that voters have private information about the intensity of own preferences, their equilibrium has some strategic delay (more on delay in Section 7), but the alternative selected in symmetric equilibrium of their model is characterized by a similar quasi-midrange condition as here. Whichever set of assumptions is more realistic in a particular context – the ones of this study, or the ones in Ponsati and Sakovics (1996) – this feature of the equilibrium outcome appears to be robust, and, consequently, the efficiency analysis contained in Section 5 looks compelling.

Other models that study collective bargaining should also be mentioned. Baron and Ferejohn (1989) presents a dynamic model of collective bargaining in which the size of supermajority is an important object of analysis. A randomly selected agent makes a proposal, which is then voted by the committee; if rejected, the game moves to the next round. This model and related literature treats the agenda as endogenous. The assumption in the model below is different – once voting starts, the alternatives considered by the voters are fixed. This assumption seems more realistic in some contexts, such as choosing a leader form a given set of candidates, if side transfers and favours among the participants are prohibited.

Compte and Jehiel (2010) study a model somewhat related to Baron and Ferejohn (1989) in which, however, the voters do not have a control over the current agenda (and thus in this respect their work is similar to the model presented below). Voters can reject the current proposal expecting that a new proposal – which arrives from outside, like in the classical search literature – will be superior.

Piketty (2000) investigates repeated voting on one issue with the same set of voters. However, he is concerned with a two-round election with three candidates, in which the second round is reached by two best candidates of the first round, only if no candidate obtains 50% of the vote in the first round.[4]

[4] There is also a literature on sequential voting, in which one set of voters votes in one

3 Model of repeated voting

Physical environment. There is a committee consisting of $n \geq 2$ agents or *voters*. There is a set of two *alternatives*, A and B. One of them may be selected at a certain time. The *outcome* is a pair consisting of the selected alternative and the time when this decision was reached.

Voting. Decision is made via supermajority voting. In a given *voting round* each voter casts her vote for A or for B. The supermajority required to select an alternative is $n + 1 - m$. The key number $m = 1, 2, ..., \bar{m}$ is interpreted as a *minimal blocking minority*, where $\bar{m} = n/2$, if n is even, and $\bar{m} = (n + 1)/2$, if n is odd. In particular, one extreme system is unanimity, in which one voter is able to block a decision, $m = 1$; the other polar case is simple majority, in which at least a half of all voters is required to block a decision, $m = \bar{m}$.

If no alternative gathers enough support, the voting goes to the next round. There is an infinite number of voting rounds. The time interval between two consecutive voting rounds is Δ, so that voting rounds occur in calendar times $t \in \{0, \Delta, 2\Delta, ...\}$.

In each round, voters announce their votes sequentially one-by-one, according to an order determined randomly and announced at the beginning of the round.[5]

Preferences. The payoff of each voter depends on the outcome. Voters are heterogeneous; commonly known parameter x_i summarizes preferences of voter i. It is interpreted in the following way: if B is selected *immediately*, then voter i gets x_i more utility than if A was selected immediately. As a result, positive x_i indicates that i prefers alternative B; negative x_i indicates that i prefers alternative A. If x_i is zero, voter i is indifferent between alternatives. Let $x = (x_1, ..., x_n)$.

To describe how delay affects the payoff, assume that $W(|x_i|, t)$ is the payoff of a voter characterized by x_i, if her favorite alternative (B if $x_i > 0$

period and then another set of voters votes later on the same or related issue (Battaglini et al. (2007) and references therein).

[5] This assumption circumvents a coordination problem inherent in voting. If voting was simultaneous in each round, then both alternatives could be obtained as equilibrium outcomes. This is also true in static games, in which, however, one may apply an argument of weak dominance to eliminate unreasonable equilibria. This refinement is not useful in repeated voting, but still, an equilibrium which relies on pure miscoordination is deemed unreasonable – and it disappears if there is some very mild inertia in voting e.g. if actions are taken sequentially within each round.

and A if $x_i < 0$) is implemented at time t, and $L(|x_i|, t)$ is the payoff of this voter if the other alternative is selected at time t. Both W and L are continuous functions of two non-negative real numbers. Clearly, for $t = 0$ we have

$$|x_i| = W(|x_i|, 0) - L(|x_i|, 0)$$

The following assumptions should be uncontroversial. Firstly, assume that for every type x_i, functions $W(|x_i|, \cdot)$ and $L(|x_i|, \cdot)$ are strictly decreasing. This captures the fact that status quo is the least preferred alternative for all agents. Secondly, for every type x_i and time t, let $W(|x_i|, t) \geq L(|x_i|, t)$.

Next assumption captures the fact the committee is locked in a room and, therefore, player's cost of waiting is increasing to such an extent that there exists a time in the future, such that when the game reaches that time, the player prefers to obtain the less preferred alternative than to wait one more round and get the more preferred alternative. Let this time be denoted by $\tau(|x_i|, \Delta)$. It is formally defined as:

$$W(|x_i|, t + \Delta) = L(|x_i|, t) \text{ if } t = \tau(|x_i|, \Delta)$$

and $W(|x_i|, t + \Delta) > L(|x_i|, t)$ (respectively, $<$) if $t < \tau(|x_i|, \Delta)$, (respectively, if $>$). Call $\tau(|x_i|, \Delta)$ the *indifference time* of voter i.

As an example, consider $W(|x_i|, t) = |x_i| - t^2$ and $L(|x_i|, t) = -t^2$. Then

$$\tau(|x_i|, \Delta) = \frac{|x_i|}{2\Delta} - \frac{\Delta}{2}$$

if the right-hand side is non-negative, zero otherwise.

The last assumption of this section puts a very mild restriction on what happens when voting becomes more frequent.[6] It is immediate that the more frequent voting, the greater $\tau(|x_i|, \Delta)$ gets. The assumption is that, for any two types x and x', the difference $\tau(|x|, \Delta) - \tau(|x'|, \Delta) \neq 0$ does not change its sign and is bounded away from zero, as voting becomes more frequent. Clearly, this assumption holds in the example.

[6] Section 5, with normative results, has an additional assumption that τ is increasing in its first argument.

4 Equilibrium

This section shows that the model described above has a subgame perfect Nash equilibrium[7] that leads to a unique voting outcome. It is completely characterized by the indifference times of two pivotal voters. To explain who pivotal voters are, it is useful first to define τ_i as

$$\tau_i = \begin{cases} \tau(|x_i|, \Delta) & \text{if} \quad x_i > 0 \\ -\tau(|x_i|, \Delta) & \text{if} \quad x_i \leq 0 \end{cases}$$

That is, $|\tau_i|$ is an indifference time of voter i, where negative sign of τ_i indicates that i prefers A, and positive sign indicates that i prefers B. Let $\tau = (\tau_1, ..., \tau_n)$. Sort the voters from the lowest to the highest according to this parameter, $(\tau_{1:n}, ..., \tau_{n:n})$, where $\tau_{k:n}$ is the kth lowest element in τ. For a given supermajority m, the *pivotal* voters are those who are characterized by indifference times $\tau_{m:n}$ and $\tau_{n+1-m:n}$. That is, these are the mth closest voter to alternative A and the mth closest voter to alternative B, according to their indifference times.

Let $h \in \{A, B\}$ be the alternative preferred by the pivotal voter with the higher indifference time and let $l \in \{A, B\}$ be the other alternative. For example, $(h, l) = (A, B)$ if $|\tau_{m:n}| > |\tau_{n+1-m:n}|$.

The main result of this section is this:

Proposition 1 *Let $m < (n+1)/2$. Fix a preference profile $(x_1, ..., x_n)$, such that $|\tau_{m:n}| \neq |\tau_{n+1-m:n}|$. Then there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, an equilibrium exists and in any equilibrium the alternative preferred by the pivotal voter with the greater indifference time, h, is selected in the first round.*

The proofs are in the Appendix.

To explain this proposition, let us define a few objects. Let $\mathcal{N}^k$ be the set of voters who prefer alternative k. Let $T^A = |\tau_{m:n}|$ and $T^B = |\tau_{n+1-m:n}|$ be the indifference times of the pivotal voters. Obviously, $T^h = \max\{T^A, T^B\}$ and $T^h > T^l$. Having T^h, we can define sets

$$\begin{aligned} \mathcal{N}_+^k &= \left\{i \in \mathcal{N}^k : \tau(|x_i|, \Delta) \geq T^h\right\} \\ \mathcal{N}_- &= \left\{i \in \mathcal{N} : \tau(|x_i|, \Delta) < T^h\right\} \end{aligned}$$

[7] A strategy is a mapping from the realization of preferences, x, and a history of voting in the previous and the current round, into action "vote for A" or "vote for B".

There are two important observations about set $\mathcal{N}_+^k$. Firstly, every voter in set $\mathcal{N}_+^k$ still strictly prefers alternative k if voting is in round $t \in \left(T^l, T^h\right)$. Secondly, set $\mathcal{N}_+^h$ has at least m elements, and $\mathcal{N}_+^l$ has at most only $m-1$ elements. Note also that as voting becomes more frequent, the order of voters according to their indifference times does not change, nor does the identity of pivotal voters, nor do the sets $\mathcal{N}_+^A$, $\mathcal{N}_+^B$ and $\mathcal{N}_-$.

The proof is essentially by backward induction. Suppose that the game is still unresolved in a voting round $t \in \left(T^l, T^h\right)$. Voters in set $\mathcal{N}_+^h$ are *able* to block alternative l if they voted in unison, because their number is at least equal to the minimal blocking minority. Moreover, they *want* to block alternative l, if h can be selected in round t or soon after, because they still strictly prefer it. This logic does not apply to voters in set $\mathcal{N}_+^l$. There are fewer of them than the minimal blocking minority, so they are not able to block alternative h in that round. Remaining voters, who are in set $\mathcal{N}_-$, do not care which alternative is selected; they want the voting to finish as quickly as possible. So, if voters in sets $\mathcal{N}_+^h$ and $\mathcal{N}_-$ can coordinate, they would coordinate on voting for h, as together they form a sufficient supermajority to stop the game in round t and select h. The fact that voting is sequential within each round, rounds are frequent, and the order of moves is established randomly enables such a coordination. Then, backward induction, round-by-round, does the rest.

5 Efficient voting system

This is the main part of this paper, in which we ask what kind of voting system should be chosen. In particular, what m is the best? It is clear that normative analysis can be illuminating only from the ex ante perspective, when the preference profile of the committee members is not yet known.[8] Assume thus that there are two stages in which citizen-voters are active. The first stage can be called a constitution design stage, and the second stage is the actual voting process culminating with the collective decision. In the first stage, voters, still uninformed about their future preference type, decide about m. Once the voting system is determined, nature selects a preference parameter for each voter and announces it publicly. In the second stage, voters, now heterogeneous, play a repeated voting game as described in the previous section.

[8] See Buchanan and Tullock (1962), page 78.

Imagine a representative voter who forms expectation about future preferences. These preference parameters will be independently drawn from a known, continuous distribution function $F(\cdot)$, with density $f(\cdot)$, mean μ and variance $\sigma^2 < \infty$.

The assumption about independence of preferences rules out some interesting scenarios. For example, suppose that the vote is about what voter 1 should have for dinner, an Apple or a Banana. Since every other voter is indifferent about the dinner of voter 1, preferences are not independent. This situation is not covered by the investigation of this section, even if Proposition 1 suggest the answer: the only system that gives voter 1 his preferred dinner with certainty is unanimity.

If F is not continuous then two voters may have the same preference parameters or the same indifference times, so that one could not use Proposition 1.

Distributions satisfying a number of assumptions will be of interest:

1. *Symmetry* of the distribution, that is, $f(\mu - x) = f(\mu + x)$ for all x.

2. *Zero mean*, that is, $\mu = 0$,

3. Existence of almost *average voters*, that is, density f is positive and continuous at μ,

4. *Bounded support*, that is, $\tilde{x} = \sup\{x : F(x) < 1\} < \infty$

Let $X_1, ..., X_n$ be the i.i.d. random variables from this F. The realization is interpreted as the preference vector x defined above. Let also $X_{1:n}, ..., X_{n:n}$ be this sample sorted from the lowest to the highest; in other words, $X_{m:n}$ is the mth order statistic associated with the sample $X_1, ..., X_n$.

This paper adopts an equally weighted utilitarian welfare function. The realization of the sample mean $\bar{X}_n = (1/n)\sum_{i=1}^{n} X_i$ is the average welfare, if B is implemented without any delay. The average welfare is $-\bar{X}_n$ if A is implemented without any delay. Clearly, the efficient outcome will not involve any delay, so the first-best mechanism implements alternative B immediately if the realization of $\bar{X}_n$ is positive and implements A immediately otherwise. Thus, the ex post first-best average welfare is $\left|\bar{X}_n\right|$, and its ex ante expected value is

$$\Pr\left\{\bar{X}_n > 0\right\} E\left(\bar{X}_n | \bar{X}_n > 0\right) + \Pr\left\{\bar{X}_n < 0\right\} E\left(-\bar{X}_n | \bar{X}_n < 0\right)$$

Note that this average welfare level is not achievable if only mechanisms studied in this paper are available, because the equilibrium outcome of these mechanisms depends only on preferences of two pivotal voters, instead of the preferences of the entire committee.

To see what is possible, consider equilibria characterized by Proposition 1 and focus on the limit case of infinitely frequent voting $\Delta \to 0$.

Assume that $\tau(\cdot, \Delta)$ is strictly increasing; that is, the stronger opinion a voter has, the longer she is willing to wait. It is a form of symmetric time preference. In equilibrium, alternative B is selected as long as $|\tau_{m:n}| < |\tau_{n+1-m:n}|$. This condition holds if and only if $|x_{m:n}| < |x_{n+1-m:n}|$, which itself is equivalent to $x_{m:n} + x_{n+1-m:n} > 0$. From the ex ante stage, one is interested in the realization of the associated random variable $Z_n^m = (1/2)(X_{m:n} + X_{n+1-m:n})$. It is called the sample mth quasi-midrange of the sample $X_1, ..., X_n$, because it is a sample midrange after one has truncated $m-1$ most extreme sample points from each side. It could be viewed as a sample measure of centrality, with sample midrange and sample median being two polar cases.

The expected average welfare of a representative agent in equilibrium is

$$V_n^m = \Pr\{Z_n^m > 0\} E\left(\bar{X}_n | Z_n^m > 0\right) + \Pr\{Z_n^m < 0\} E\left(-\bar{X}_n | Z_n^m < 0\right) \quad (1)$$

The interest is in the second-best mechanisms, obtained by finding m that maximizes V_n^m. Such m will be referred to as *constrained efficient supermajority.*[9]

In order to study efficiency properties of various supermajority rules, one has to start by examining the joint probability distribution of $\left(\bar{X}_n, Z_n^m\right)$. It is easy to calculate if $n = 2$, (because then $\bar{X}_2 = Z_2^2$) although that exercise is useless if one wants to study supermajority rules. Calculating this probability distribution explicitly becomes difficult for $n = 3$; for all practical reasons it is impossible if n is greater. A systematic study of V_n^m as a function of m looks like a hopeless task.

The rest of this section focuses on the case $n \to \infty$. Fortunately, it turns out that the resulting asymptotic joint distribution has a very tractable form and allows a far-reaching explicit analysis of constrained efficient mechanisms. Hence, all normative statements in this section will be double limit results. Firstly, the time interval between voting rounds goes to zero, and then the size of the committee goes to infinity.

[9]The second efficiency criterion that one may use is the probability of the correct choice, $P_n^m = \Pr\left\{\bar{X}_n \times Z_n^m > 0\right\}$. Rae (1969) used this measure.

5.1 Asymptotic joint probability distribution

Switch from an absolute measure of supermajority, m, to a relative measure, $p \in (0, 1/2]$, a fraction of all the voters that may block an alternative. They are related through $m = \lceil np \rceil$. In other words, as n goes to infinity, the supermajority requirement m also goes to infinity, but $p = m/n$ stays constant (subject to an integer constraint).

Before the key asymptotic result linking the sample mean and the sample quasi-midrange is presented, one has to define a few objects. Let $0 < p < 1$ and let x_p denote pth population[10] quantile of F, that is, $F(x_p) = p$. Apart from the sample mean $\bar{X}_n$, define also $Y_n^p = X_{\lceil np \rceil :n}$ to be the sample pth quantile. Moreover, for $0 < p < q < 1$ define $\hat{Z}_n^{pq} = (1/2)\left(Y_n^p + Y_n^q\right)$ to be the sample $(p, q)-$quasi-midrange. Let $\mu_{pq} = (1/2)\left(x_p + x_q\right)$ be the corresponding population quasi-midrange.

Given distribution F, define the following variance-covariance terms,

$$\begin{aligned} \sigma_{px} &= p(1-p)\left(E\left(X|X > x_p\right) - E\left(X|X < x_p\right)\right)/f\left(x_p\right) \\ \left(\sigma_p\right)^2 &= p(1-p)/f\left(x_p\right)^2 \end{aligned}$$

For $0 < p < q < 1$ define also

$$\begin{aligned} \sigma_{pq} &= p(1-q)/\left(f\left(x_p\right)f\left(x_q\right)\right) \\ \sigma_{xz} &= (1/2)\left(\sigma_{px} + \sigma_{qx}\right) \\ \left(\sigma_z\right)^2 &= (1/4)\left(\sigma_p^2 + 2\sigma_{pq} + \sigma_q^2\right) \end{aligned}$$

The following result reveals the asymptotic normality of the sample mean and a sample quasi-midrange, as the sample size goes to infinity.

Lemma 1 *Let $0 < p < q < 1$. If density f is continuous and positive at x_p and x_q, then*

$$\sqrt{n}\left(\begin{bmatrix} \bar{X}_n \\ \hat{Z}_n^{pq} \end{bmatrix} - \begin{bmatrix} \mu \\ \mu_{pq} \end{bmatrix}\right) \xrightarrow{d} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & \sigma_{xz} \\ \sigma_{xz} & \left(\sigma_z\right)^2 \end{bmatrix}\right)$$

This is a general result, but in the context of this model, one is interested in the case when $p = 1 - q \leq 1/2$. This is because both alternatives are

[10] The term "population" refers to the corresponding parameter of the actual distribution function. The term "sample" refers to the realization of a certain random variable and represents the population of voters.

treated symmetrically by the voting system, that is, each must gather the same support to win. Note that under this assumption, $\hat{Z}_n^{1-q,q} = \hat{Z}_n^{p,1-p} = Z_n^m$ as long as $m = \lceil np \rceil$.[11] Abusing notation slightly, write Z_n^q for Z_n^m, write V_n^q for V_n^m.

The asymptotic correlation between the sample mean and the sample quasi-midrange will be a key object of analysis. It is defined by

$$\rho(x_q) = \frac{\sigma_{xz}}{\sigma\sigma_z}, \tag{2}$$

where its dependence on the quantile determined by the voting system q is emphasized by the notation. Let $R(x) = E(X - x | X > x)$ be the *mean residual life* associated with distribution F (this function is often used in reliability studies).

Lemma 2 *Suppose that F satisfies assumption 1 (symmetry), and the density f is continuous and positive at x_q. Then the asymptotic correlation coefficient of the mean and the qth quasi-midrange is*

$$\rho(x_q) = \frac{\sqrt{2}}{\sigma}\sqrt{1 - F(x_q)}\left(R(x_q) + x_q - \mu\right) \tag{3}$$

for $x_q \geq \mu$. Its derivative is

$$\rho'(x_q) = \frac{1}{\sigma\sqrt{2}}\frac{f(x_q)}{\sqrt{1 - F(x_q)}}\left(R(x_q) - x_q + \mu\right). \tag{4}$$

5.2 Welfare criterion

Having Lemma 1, the asymptotic constrained efficiency criterion is easy to derive.

Lemma 3 *Suppose that F satisfies assumptions 1 (symmetry) and 2 (zero mean), and the density f is continuous and positive at x_q. Then the asymptotic welfare is $\lim_{n\to\infty}\sqrt{n}V_n^q = \rho(x_q)\,\sigma\sqrt{2/\pi}$.*

[11] Strictly speaking, this is correct if np is not an integer.

Thus, the task of maximizing welfare is equivalent to the task of maximizing $\rho(x_q)$.[12,13]

5.3 Efficient supermajority rule

This subsection uses the above results to characterize the constrained efficient supermajority rule.

First, investigate a system close to simple majority. To do that, check the sign of $\rho'(\cdot)$, when x_q approaches the median $\mu = 0$ from the right. Lemma 2 shows that the derivative of the correlation becomes $f(\mu) R(\mu)/\sigma$. This term remains strictly positive, because voters with non-average preferences may exist, $R(\mu) > 0$, and because voters with average preferences may exist, $f(\mu) > 0$. Asymptotic correlation is strictly increasing at μ, so is the asymptotic average welfare, by Lemma 3. This proves

Proposition 2 *Suppose that F satisfies assumptions 1 (symmetry), 2 (zero mean) and 3 (average voters). Then the asymptotic correlation is a strictly increasing function at μ, and simple majority is not asymptotically constrained efficient.*

Next, focus on the other polar case – supermajority rules close to unanimity. Straightforward examination of equation (3) shows that $\lim_{q\to 1} \rho(x_q)$ is zero, if the distribution of preferences has a bounded support. Then Lemma 3 implies the following result,

Proposition 3 *Suppose that F satisfies assumptions 1 (symmetry), 2 (zero mean) and 4 (bounded support). Then* $\lim_{q\to 1} \lim_{n\to\infty} \sqrt{n} V_n^q = 0$.

Asymptotic correlation is a continuous function of supermajority q, starting at a certain strictly positive value for $q = 1/2$, then increasing as supermajority requirement is increased (by Proposition 2), but eventually turning

[12] It appears that the value of welfare converges to zero as $n \to \infty$, and so the result of voting in large committees is irrelevant to an individual. This is an incorrect interpretation, see discussion in Section 6.1.

[13] The asymptotic probability of the correct choice is

$$\lim_{n\to\infty} P_n^q = 1/2 + (1/\pi) \arcsin(\rho(x_q))$$

which has the same maximizer as $\lim_{n\to\infty} \sqrt{n} V_n^q$.

down so dramatically that it reaches zero (by Proposition 3). This proves that a constrained efficient supermajority rule must be strictly in the interior. The first order necessary condition is $\rho'(x_q) = 0$, which translates to $R(x_q) = x_q$. We obtain

Proposition 4 *Suppose that F satisfies assumptions 1 (symmetry), 2 (zero mean) and 4 (bounded support). Suppose that density f is positive and continuous on support $(-\tilde{x}, \tilde{x})$. Then constrained efficient q is in the interior, $1/2 < q < 1$, and satisfies $R(x_q) = x_q$.*

Writing it concisely: under all assumptions of the model above, the ranking of the supermajority systems according to their welfare performance is:

$$\text{Unanimity} \prec \text{Simple majority} \prec \text{Optimal interior supermajority}$$

As the last remark in this subsection, we investigate unanimity using a different type of asymptotic analysis. Suppose that m is constant when $n \to \infty$. In this case, $p = m/n$ converges to zero, representing the case of an extreme supermajority requirement and, in particular, the case of unanimity. There is a known result, which states that three random variables $X_{m:n}$, $X_{n+1-m:n}$ and $\sqrt{n}\left(\bar{X}_n - \mu\right)/\sigma$ are asymptotically *independent*, as $n \to \infty$ for a given m (see for example David (1981), p. 269-270). Therefore, any quasi-midrange Z_n^m and standardized mean $\sqrt{n}\left(\bar{X}_n - \mu\right)/\sigma$ are asymptotically independent for a given m. Equation (1) together with this observation implies the following statement:

Proposition 5 *Suppose that F satisfies assumption 2 (zero mean). Suppose that m is constant. Then* $\lim_{n\to\infty} \sqrt{n} V_n^m = 0$.

This statement has a similar conclusion to Proposition 3, although under different assumptions.

5.4 Examples

Further intuition will be built by the following examples.

5.4.1 Bounded support: Piecewise linear distribution

Consider a piecewise linear distribution function with the support $[-1,1]$, symmetric around the mean zero. It is characterized by parameters α and β, where $\beta \in [0,1]$ is the cutoff level on the support, and $\alpha \in (0,1)$ is the probability that $|x| \in (\beta, 1)$. That is, the functions $F(x)$ and $R(x)$ for positive arguments are:

Interval	$F(x)$	$R(x)$
$0 \leq x \leq \beta$	$\frac{1}{2} + \frac{(1-\alpha)}{2\beta}x$	$\frac{\alpha(1+\beta)\beta+(1-\alpha)(\beta-x)(\beta+x)}{2(\lambda\beta+(1-\alpha)(\beta-x))} - x$
$\beta < x \leq 1$	$1 - \frac{\alpha}{2(1-\beta)}(1-x)$	$\frac{1+x}{2} - x$

First, consider a uniform distribution on $[-1,1]$, which requires $\beta = 1-\alpha$. Then $R(x) = \frac{1+x}{2} - x$. The optimality condition, $R(x) = x$, leads to the unique solution at $x = 1/3$. The supermajority system that supports this quantile is $q = 2/3$.

Now, let us ask the question the other way round: what is the distribution, if an arbitrary $q \in (1/2, 1)$ is to be constrained efficient? The following statement provides the answer.

Proposition 6 *Select any $q \in (1/2, 1)$. Let a piecewise linear distribution have parameters*

$$\alpha = 3\frac{1-q}{2-q} \text{ and } \beta = \frac{1-q}{2-q}$$

Then, q is the constrained efficient supermajority rule.

Note that a very extreme supermajority requirement may be optimal. As $q \to 1$, both parameters converge to zero. The intuition is clear: if almost everyone in a committee is almost indifferent between the alternatives, then very few opinionated voters should be allowed to decide about the outcome. That means a very high supermajority requirement.

On the other hand, simple majority may also be optimal. We have $\lim_{q\to 1/2} \beta = 1/3$ and $\lim_{q\to 1/2} \alpha = 1$. That is, a simple majority is optimal only if all voters prefer their favorite alternatives with similar strengths, and, in particular, only if no one is close to being indifferent.

5.4.2 Unbounded support: Pareto distribution

Consider a family of two-sided generalized Pareto distributions, symmetric around zero. If x is positive, the c.d.f. function is

$$F(x) = 1 - \frac{1}{2}\left(\frac{b}{ax+b}\right)^{1+1/a}$$

where the parameters satisfy $b > 0$ and $a > 0$. The variance exists if and only if $a < 1$. The case of $a = 0$ is described by the exponential distribution with parameter b:

$$F(x) = 1 - \frac{1}{2}\exp\left(-\frac{x}{b}\right)$$

Pareto distribution is easy to work with because it generates a linear mean residual life function, $R(x) = ax + b$. This paper focuses only on parent distributions for which the first two moments exist, so that our earlier asymptotic results can be derived. As a consequence, we reach conclusion that for all parameters in the interesting range, $b > 0$ and $a \in (0,1)$, there is an interior solution to the condition $R(x_q) = x_q$. The optimal quantile is $x_q = b/(1-a)$ and the constrained efficient supermajority is

$$q = 1 - \frac{1}{2}(1-a)^{1+1/a}$$

As a goes to zero, the distribution becomes exponential, and the optimal supermajority becomes $1 - (1/2)\exp(-1)$, which is roughly equal to 0.816. Alternatively, as a goes to one, the tails become thicker and we obtain that large supermajority are optimal, $\lim_{a\to 1} q = 1$.

Unresolved question remains whether this can be generalized to all unbounded distributions. That is, is it true that for any distribution having a finite second moment, condition $R(x) < x$ holds for all x sufficiently high? If yes, then there is an interior solution. The following results gives an answer to a related question.

Proposition 7 *Suppose $\tilde{x} = \infty$. Then there exists $x \in (0,\infty)$ such that $R(x) = x$.*

The second unresolved question is what happens if $\sigma^2 = \infty$? Neither conclusion of Lemma 1, nor of the result leading to Proposition 5 is valid in this case. One can conjecture that if tails of the distribution are thick enough, preferences of the two most extreme voters would become so important that in the efficiency calculation they would dominate everyone in between. If that is true, then unanimity would be the best system.

6 Criticisms

This section will address some of the criticisms that can be raised against the asymptotic efficiency analysis above.

6.1 Is the result of voting important to an individual?

The fact that $\sqrt{n}V_n^q$ converges to a constant means that V_n^q converges to zero, and, therefore, the result of voting in a large committee appears to be irrelevant to an individual.

Such a logic is incorrect. This paper studies comparative statics of the size of the supermajority for a given size of the committee, and not of the size of the committee. It just turns out that this former question has a very clean answer if the committee is large. We cannot say that outcomes in bigger committees are less important to an individual than in smaller committees. As an example, suppose that true preference value is $c_n x_i$, where, as before, x_i comes from a known distribution, and where c_n depends only on the size of the committee n. That is, the stakes to an individual are related to the size (or fineness) of the committee. In this case, the conclusion would be that $\left(\sqrt{n}/c_n\right) V_n^q$ converges to a constant, and hence the result of voting in a large committee is explosively more important to an individual, as long as c_n grows faster than $\sqrt{n}$, as $n \to \infty$. The key fact is that regardless of what c_n is, the analysis of the efficient supermajority in the limit is unaffected and therefore in order to investigate the question asked in this paper one can just assume that $c_n = 1$.

Moreover, an alternative indicator can be used as an efficiency criterion – the probability of the correct choice, as defined in footnote 13. The implications are exactly the same, but this measure does not vanish as the size of the committee goes to infinity.

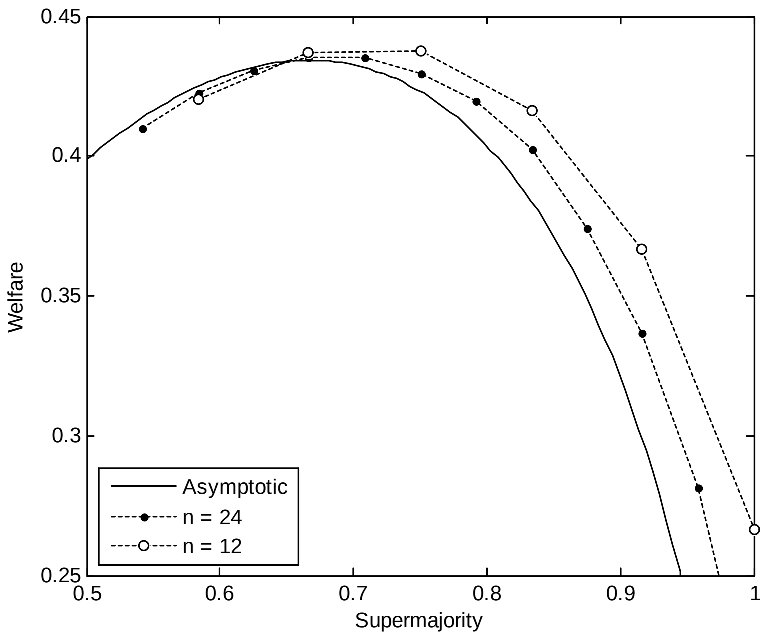

Figure 1: Welfare (uniform distribution)

6.2 Is the asymptotic case a good approximation of moderately sized committees?

Real-life committees have moderate sizes: hiring committees may have fewer than 10 members, juries are often composed of 12 jurors, papal conclave has up to 120 Cardinals, but historically this number has been much lower. Is the asymptotic case a reasonable approximation of a committee with a moderate number of voters?

This evaluation is done by means of a simulation. Figure 1 shows a case in which individual preference parameters come from the uniform distribution. The horizontal axis shows the supermajority size, and the vertical axis measures expected welfare scaled by $\sqrt{n}$. Three cases are shown: committee consisting of 12 members, 24 members and the asymptotic case.

All three cases generate welfare levels that are remarkably similar. Note that moderate-sized committees are inherently coarse, e.g. for $n = 12$, all supermajority requirements $q \in (6/12, 7/12]$ imply that 7 out of 12 members must agree for an alternative to be selected. They all generate expected welfare levels 0.415, represented by the first empty dot on Figure 1. What is more important is that these three cases result in a similar welfare-maximizing supermajority, close to the asymptotic case 2/3. This example suggests that the asymptotic case is a reasonable guide in trying to assess the welfare-maximizing supermajority even if committees are relatively small.

This also suggest that even if the distribution is not perfectly but only somewhat symmetric or the mean of the distribution is close but not exactly zero, the asymptotic case may serve as a convenient first approximation.

6.3 What if preferences are correlated?

Next, consider an asymmetric distribution of preferences. This case can emerge naturally when analyzing voting in jury trials, for instance. Apart from some private component, jury trials have a strong common component indicating that all voters agree about objective guilt or innocence of the accused; in other words, individual preferences are somewhat positively correlated. One can model this in the following way: after supermajority is decided, the nature selects the value of a parameter μ, which can be equal to either ν, or $-\nu$ with equal probabilities, and then selects and announces a profile of types from a distribution F which is symmetric around μ. For $\nu = 0$, this model is equivalent to the one studied above; the further away

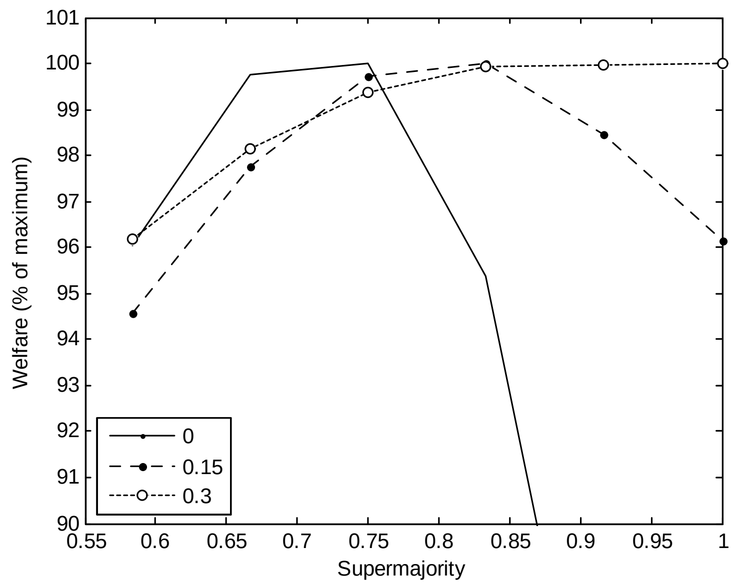

Figure 2: Welfare for different ν (uniform distribution and $n = 12$).

from zero ν is, the stronger the agreement among the voters.

Clearly, for ν large enough (and F with bounded support) there is no disagreement among the voters about which alternative is the best and hence any supermajority level selects the efficient outcome. In contrast to this extreme case, Figure 2 illustrates the welfare for three levels of moderate ν (this is based on $n = 12$ voters and F uniform around μ). The baseline scenario is $\nu = 0$ with efficient supermajority $9/12$. For $\nu = 0.15$ the efficient supermajority is $10/12$, and for $\nu = 0.3$ it is $12/12$. This example suggests that, ceteris paribus, an efficient supermajority may even be larger in a case in which preferences are correlated than when they are independent around $\mu = 0$.

7 Final remarks

This paper studies conclaves – voting procedures in which one of the alternatives has to be approved by some supermajority, and hence voting may be repeated many times before the decision is made. The main conclusion of the analysis is this: if voters' preference intensities differ, then a conclave with some intermediate level of supermajority should increase efficiency relative to simple majority. Simple majority gives all players the same vote, regardless of whether they are indifferent or extreme. Supermajority, on the other hand, gives ultimate power to voters who are not indifferent, and this may improve efficiency.

Technically, this is shown in two steps. Firstly, it is observed that the sample mean may correlate with some intermediate sample quasi-midranges better than with the sample median. Secondly, it shows that the equilibrium in repeated voting with supermajority depends on the "quasi-midrange" voter. Combining these two observations, one can design a constrained efficient supermajority rule, which maximizes the correlation between the equilibrium outcome, characterized by the quasi-midrange, and the first-best efficient outcome, characterized by the mean. Under assumptions of Section 5, simple majority is better than unanimity, but both are worse than some intermediate supermajority.

Finally, delay does not play a role in efficiency calculation in this model. The fact that there is no delay in reaching the decision in equilibrium of Proposition 1 is a consequence of a number of assumptions. Common knowledge of payoffs is an important one. If it was assumed instead that x_i was known only to voter i (Ponsati and Sakovics (1996)), then delay would typically occur and would serve as a mechanism sorting indifferent voters from the zealous ones. This delay implies an efficiency loss, which, if high enough, could eat up all the benefits that better voting decisions bring.

However, it is worth pointing out that even in such a model, where delay does occur in equilibrium, it may be reasonable to ignore it in efficiency calculation. Often, the committee members represent a wider population and so the quality of their decision may have an external effect on this population (such as recruitment committee representing an organization, or jury members working in the interests of the society, or in emergency negotiation marathons where negotiators represent their political parties, countries or organizations). By increasing the cost of waiting (locking the committee members indefinitely in a room only with low-quality coffee), one can shorten the equilibrium waiting time of the wider population, although the cost of delay paid by the committee members in symmetric equilibrium is not changed (essentially by the Revenue Equivalence Theorem). In the extreme, if the population is infinitely larger than the committee representing it, the only factor in efficiency calculation ought to be the quality of the decision, and not the delay suffered by the committee members.

A Appendix

A.1 Proof of Proposition 1.

We first characterize an equilibrium outcome; existence then follows.

Fix n, m and $(x_1, ..., x_n)$. Let h be the alternative supported by the pivotal voter with a greater indifference time, and let l be the other alternative, $T^h > T^l$. Let $\hat{t}^\Delta(t)$ be the first voting round after calendar time t, and let $\check{t}^\Delta(t)$ be the last voting round before calendar time t, given the time grid $\{0, \Delta, 2\Delta, ...\}$.

Lemma 4 *There exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, the following is true: if the game reaches a voting round $\hat{t}^\Delta(T^l) \in (T^l, T^h)$, then in any equilibrium the game stops at $\hat{t}^\Delta(T^l)$ and alternative h is selected.*

Proof. Choose $\bar{\Delta}$ small enough, so that there are at least two distinct rounds in interval (T^l, T^h).

1. There is $\eta > 0$, such that in any round $t \in (T^l, T^h)$ alternative h is selected with probability at least η.

   Suppose the game reaches a round $t \in (T^l, T^h)$. Voters are ordered randomly before any round. Let η be a probability that they are ordered such that all voters in set $\mathcal{N}_-$ take decision after all voters in set $\mathcal{N}_+^h$, in which case alternative h is selected at round t. The proof of this last statement is by backward induction within round t. In particular, note that if all voters in $\mathcal{N}_+^h$ vote for h, then any voter in $\mathcal{N}_-$ realizes that l cannot be implemented in round t. If voters in $\mathcal{N}_-$ vote for h as well, they guarantee the best outcome for themselves, that the game finishes in round t. Alternative h is implemented in round t regardless of what voters in set $\mathcal{N}_+^l$ do.

2. For any $\varepsilon > 0$, there is $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$ we have: in round $t = \hat{t}^\Delta(T^l)$, any player $i \in \mathcal{N}_+^h$ has the minimum equilibrium payoff $(1-\varepsilon) W(|x_i|, t) + \varepsilon L(|x_i|, t)$.

   Firstly, observe that the interval (T^l, T^h) shifts as Δ gets smaller. However, by assumption, its size is bounded away from zero, and hence the number of rounds in it grows to infinity.

In every round, players in set $\mathcal{N}_{+}^{h}$ have an independent chance of getting a good payoff. To be more specific, consider two dates t and $t-\Delta$ in $\left(T^{l}, T^{h}\right)$ and let $\tilde{L}\left(\left|x_i\right|, t\right)$ be a lower bound on payoff of a voter $i \in \mathcal{N}_{+}^{h}$ at the beginning of a voting round t (or at the end of round $t-\Delta$). At the beginning of round $t-\Delta$, nature decides about the order of voters in that round. With probability η the order is as in point 1, implying payoff $W\left(\left|x_i\right|, t-\Delta\right)$. With remaining probability, the order is different, but the payoff to i is no less than $\tilde{L}\left(\left|x_i\right|, t\right)$. This is because otherwise all voters in $\mathcal{N}_{+}^{h}$ have a common interest to vote for h in order to either obtain h in this round, or send the game to the next round t where payoff at least $\tilde{L}\left(\left|x_i\right|, t\right)$ is obtained. So, a lower bound on payoff at the beginning of period $t-\Delta$ is

$$\tilde{L}\left(\left|x_i\right|, t-\Delta\right)=\eta W\left(\left|x_i\right|, t-\Delta\right)+(1-\eta) \tilde{L}\left(\left|x_i\right|, t\right) \tag{A.1}$$

Now, select $\varepsilon>0$, as desired. Select a calendar time $t' \in\left(T^{l}, T^{h}\right)$ close enough to $t=T^{l}$, so that

$$W\left(\left|x_i\right|, t'\right)>(1-\varepsilon) W\left(\left|x_i\right|, t\right)+\varepsilon L\left(\left|x_i\right|, t\right) \tag{A.2}$$

Since $W\left(\left|x_i\right|, \cdot\right)$ is decreasing and continuous, such t' exists. Let $\zeta=\left(\check{t}^{\Delta}\left(t'\right)-\hat{t}^{\Delta}(t)\right) / \Delta$ be the number of successive voting rounds between rounds $\hat{t}^{\Delta}(t)$ and $\check{t}^{\Delta}\left(t'\right)$. The bound on lowest payoff in round $\check{t}^{\Delta}\left(t'\right)$ is $L\left(\left|x_i\right|, t'\right)$. Taking equation (A.1) and applying it recursively ζ times back to round $\hat{t}^{\Delta}(t)$ we obtain

$$\tilde{L}\left(\left|x_i\right|, \hat{t}^{\Delta}(t)\right) \geq\left(1-(1-\eta)^{\zeta}\right) W\left(\left|x_i\right|, t'\right)+(1-\eta)^{\zeta} L\left(\left|x_i\right|, t'\right) \tag{A.3}$$

Select $\bar{\zeta}$ large enough so that

$$(1-\eta)^{\bar{\zeta}}<\frac{W\left(\left|x_i\right|, t'\right)-\left((1-\varepsilon) W\left(\left|x_i\right|, t\right)+\varepsilon L\left(\left|x_i\right|, t\right)\right)}{W\left(\left|x_i\right|, t'\right)-L\left(\left|x_i\right|, t'\right)}$$

Such a $\bar{\zeta}$ exists because the right-hand side is positive, by A.2. Hence for all $\zeta>\bar{\zeta}$,

$$\left(1-(1-\eta)^{\zeta}\right) W\left(\left|x_i\right|, t'\right)+(1-\eta)^{\zeta} L\left(\left|x_i\right|, t'\right)>(1-\varepsilon) W\left(\left|x_i\right|, t\right)+\varepsilon L\left(\left|x_i\right|, t\right) \tag{A.4}$$

Equations (A.3) and (A.4) imply the result.

3. There is $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$ alternative h is chosen at round $t = \hat{t}^{\Delta}\left(T^{l}\right)$.

   Observe that alternative l is not selected in this round, because voters in $\mathcal{N}_{+}^{h}$ would vote against, by point 2, since for any $i \in \mathcal{N}_{+}^{h}$ we have $L\left(\left|x_i\right|, t\right) < (1-\varepsilon) W\left(\left|x_i\right|, t+\Delta\right) + \varepsilon L\left(\left|x_i\right|, t+\Delta\right)$ for ε small enough. Players in $\mathcal{N}_{-}$ prefer alternative h to be voted in round t rather than later. Given the common interest of voters in set $\mathcal{N}_{+}^{h} \cup \mathcal{N}_{-}$ and the sequential nature of voting game, alternative h is selected in round t.

■

Lemma 5 *There exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, the following is true: if alternative h is selected in round $t' > 0$, then it is selected in round $t = t' - \Delta$.*

Proof. There are three outcomes that can occur in round $t = t' - \Delta$: alternative h is selected in time t, alternative h is selected at time t' and alternative l is selected at time t. These outcomes generate payoff levels for player $i \in \mathcal{N}_{+}^{h}$ equal to $W\left(\left|x_i\right|, t\right)$, $W\left(\left|x_i\right|, t'\right)$ and $L\left(\left|x_i\right|, t\right)$, respectively. Note that if Δ is small enough, these outcomes can be ranked as $W\left(\left|x_i\right|, t\right) > W\left(\left|x_i\right|, t'\right) > L\left(\left|x_i\right|, t\right)$.

All voters are put in a sequence at the beginning of round t. Let ι index all voters who are in set $\mathcal{N}_{+}^{h}$, so that $\iota = 1$ is the last such voter, and $\iota = N_{+}^{h}$ is the first such voter.

Assume that an inductive hypothesis holds for ι: "if all voters $i \in \mathcal{N}_{+}^{h}$ who are before and up to ι in the sequence, $i = N_{+}^{h}, ..., \iota$, voted for h, then alternative l is not selected in round t". Note that if $\iota = 1$, then the inductive hypothesis is trivially satisfied, since if all voters in $\mathcal{N}_{+}^{h}$ voted for h, alternative l does not have enough support to be selected in round t. The rest of this proof will show that then the following is true "if all voters $i \in \mathcal{N}_{+}^{h}$ who are before and up to $\iota + 1$ in the sequence, $i = N_{+}^{h}, ..., \iota + 1$, voted for h, then alternative l is not selected in round t".

Consider voter ι and suppose that all voters $i \in \mathcal{N}_{+}^{h}$ who are before and up to $\iota + 1$ in the sequence, $i = N_{+}^{h}, ..., \iota + 1$, voted for h.

1. Suppose that all votes registered so far in this round are such that h can still be selected in this round t. Then in equilibrium h will be selected in this round t.

To see this, note that voter ι can vote for h and prevent l being selected in round t, by inductive hypothesis. All successive voters have strictly higher payoff from finishing the game with alternative h at t, rather than at t'. They vote so that h is selected at t. Consequently, this action of ι leads to the highest possible payoff $W(|x_\iota|, t)$.

2. Suppose that all votes registered so far in this round are such that h cannot be selected at t. Then h will be selected at time t'.

   To see this, note that voter ι can vote h and prevent l from being selected at time t, by inductive hypothesis. Payoff from this action is the highest possible $W(|x_\iota|, t')$ (since $W(|x_\iota|, t)$ cannot be achieved).

In any case, before ι takes her action, alternative l will not occur in equilibrium at round t, proving the inductive step.

Consider now all voters who take actions before voter $\iota = N_+^h$; Since alternative l will not be selected, they vote so that alternative h is selected at t. ■

The proof of the statement in Proposition 1 that the equilibrium outcome is unique is by backwards induction, voting round by voting round: the first lemma provides a starting point, and the second lemma provides a recursive step backwards.

Equilibrium existence is a by-product of this proof. For example, all agents voting for h in every round and for every history is an equilibrium.

A.2 Proof of Lemma 1

Recall the following two known results:

- The joint asymptotic distribution of the sample mean and sample quantile (Ferguson (1999)):

$$\sqrt{n}\left(\left[\begin{array}{c}\bar{X}_n\\ Y_n^p\end{array}\right]-\left[\begin{array}{c}\mu\\ x_p\end{array}\right]\right)\xrightarrow{d} N\left(\left[\begin{array}{c}0\\ 0\end{array}\right],\left[\begin{array}{cc}\sigma^2 & \sigma_{px}\\ \sigma_{px} & (\sigma_p)^2\end{array}\right]\right)$$

- The joint asymptotic distribution of two sample quantiles (for example: David (1981) Theorem 9.2):

$$\sqrt{n}\left(\left[\begin{array}{c}Y_n^p\\ Y_n^q\end{array}\right]-\left[\begin{array}{c}x_p\\ x_q\end{array}\right]\right)\xrightarrow{d} N\left(\left[\begin{array}{c}0\\ 0\end{array}\right],\left[\begin{array}{cc}(\sigma_p)^2 & \sigma_{pq}\\ \sigma_{pq} & (\sigma_q)^2\end{array}\right]\right)$$

A joint asymptotic distribution of $\left[\bar{X}_n, Y_n^p, Y_n^q\right]^T$ is normal too. To see this, follow the steps of Ferguson (1999) in deriving the asymptotic distribution of the sample mean and two sample quantiles and then apply the Cramér-Wold Theorem to show the asymptotic joint normality.

An immediate implication of the above observation is that

$$\sqrt{n}\left(\left[\begin{array}{c}\bar{X}_n\\ Y_n^p\\ Y_n^q\end{array}\right]-\left[\begin{array}{c}\mu\\ x_p\\ x_q\end{array}\right]\right)\xrightarrow{d} N\left(0_{3\times1},\Sigma\right)$$

where the covariance matrix is

$$\Sigma=\left[\begin{array}{ccc}\sigma^2 & \sigma_{px} & \sigma_{qx}\\ \sigma_{px} & \left(\sigma_p\right)^2 & \sigma_{pq}\\ \sigma_{qx} & \sigma_{pq} & \left(\sigma_q\right)^2\end{array}\right]$$

Next, apply the affine transformation to the random variables in the above convergence, with the transformation matrix

$$D=\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1/2 & 1/2\end{array}\right]$$

We then obtain

$$\sqrt{n}\left(\left[\begin{array}{c}\bar{X}_n\\ \hat{Z}_n^{pq}\end{array}\right]-\left[\begin{array}{c}\mu\\ \mu_{pq}\end{array}\right]\right)\xrightarrow{d} N\left(0_{2\times1},D\Sigma D^T\right)$$

where the covariance matrix is

$$D\Sigma D^T=\left[\begin{array}{cc}\sigma^2 & \frac{1}{2}\left(\sigma_{px}+\sigma_{qx}\right)\\ \frac{1}{2}\left(\sigma_{px}+\sigma_{qx}\right) & \frac{1}{4}\left(\sigma_p^2+2\sigma_{pq}+\sigma_q^2\right)\end{array}\right]$$

A.3 Proof of Lemma 2

Let

$$\begin{aligned}\bar{\omega}\left(x\right)&=E\left(X|X<x\right)\\ \omega\left(x\right)&=E\left(X|X>x\right)\end{aligned}\tag{A.5}$$

The asymptotic covariance between the sample mean and the sample (p,q)–quasi-midrange is

$$\sigma_{xz}=\frac{1}{2}\left(\frac{p\left(1-p\right)}{f\left(x_p\right)}\left(\omega\left(x_p\right)-\bar{\omega}\left(x_p\right)\right)+\frac{q\left(1-q\right)}{f\left(x_q\right)}\left(\omega\left(x_q\right)-\bar{\omega}\left(x_q\right)\right)\right)$$

Facts about conditional expectation are

$$q\omega\left(x_{1-q}\right)+(1-q)\,\bar{\omega}\left(x_{1-q}\right)=\mu=(1-p)\,\omega\left(x_p\right)+p\bar{\omega}\left(x_p\right)$$

Using these equations to eliminate $\omega\left(x_p\right)$ and $\bar{\omega}\left(x_q\right)$ results in

$$\sigma_{xz}=\frac{1}{2}\left(\frac{p}{f\left(x_p\right)}\left(\mu-\bar{\omega}\left(x_p\right)\right)+\frac{(1-q)}{f\left(x_q\right)}\left(\omega\left(x_q\right)-\mu\right)\right)$$

Now, recall that $p=1-q\leq 1/2$. This simplifies the covariance to

$$\sigma_{xz}=\frac{1}{2}(1-q)\left(\frac{1}{f\left(x_{1-q}\right)}\left(\mu-\bar{\omega}\left(x_{1-q}\right)\right)+\frac{1}{f\left(x_q\right)}\left(\omega\left(x_q\right)-\mu\right)\right)$$

Evoke assumption 1, about the symmetry of the distribution around the mean. For every q we have $f\left(x_{1-q}\right)=f\left(x_q\right)$; moreover, we have $\mu=\left(\omega\left(x_q\right)+\bar{\omega}\left(x_{1-q}\right)\right)/2$. Eliminate $\bar{\omega}\left(x_{1-q}\right)$ to obtain

$$\sigma_{xz}=\frac{1-q}{f\left(x_q\right)}\left(\omega\left(x_q\right)-\mu\right) \tag{A.6}$$

Likewise, the asymptotic variance of the quasi-midrange simplifies to

$$\left(\sigma_z\right)^2=\frac{1}{4}\left(\sigma_p^2+2\sigma_{pq}+\sigma_q^2\right)=\frac{1}{2}\frac{1-q}{f\left(x_q\right)^2} \tag{A.7}$$

The first part of the proposition comes from the definition of the correlation coefficient in equation (2) and equations (A.6) and (A.7).

To obtain the second part take the derivative of $\rho(x)$

$$\rho'(x)=\frac{\sqrt{2}}{\sigma}\frac{-f(x)}{2\sqrt{1-F(x)}}\left(\omega(x)-\mu\right)+\sqrt{1-F(x)}\omega'(x)$$

Note that

$$\begin{aligned}\omega'(x) &= \frac{d}{dx}\left(\int_x^{\infty}tf(t)\,dt\frac{1}{1-F(x)}\right)=-xf(x)\frac{1}{1-F(x)}+\int_x^{\infty}tf(t)\,dt\frac{f(x)}{(1-F(x))^2}\\ &= \frac{f(x)}{1-F(x)}\left(\omega(x)-x\right)\end{aligned}$$

That is

$$\rho'(x)=\frac{1}{\sigma\sqrt{2}}\frac{f(x)}{\sqrt{1-F(x)}}\left(\mu+R(x)-x\right)$$

A.4 Proof of Lemma 3

Firstly, find $E\left(\bar{X}_n|\hat{Z}_n^{pq} > 0\right)$. By Greene (2002), Theorem 22.5 we have

$$\begin{aligned}\sqrt{n}E\left(\bar{X}_n|\hat{Z}_n^{pq} > 0\right) &= E\left(\sqrt{n}\bar{X}_n|\sqrt{n}\hat{Z}_n^{pq} > 0\right) \\ &\to \rho\sigma\frac{\phi(0)}{1-\Phi(0)}\end{aligned}$$

where ϕ and Φ are the p.d.f. and c.d.f. of a standard normal distribution. Likewise

$$\sqrt{n}E\left(\bar{X}_n|Z_n^{pq} < 0\right) \to -\rho\sigma\frac{\phi(0)}{\Phi(0)}$$

Then, the average expected efficiency of a supermajority with q is

$$\sqrt{n}V_n^m \to 2\rho\sigma\phi(0) = \rho\sigma\sqrt{2}\frac{1}{\sqrt{\pi}}$$

As far as the result in footnote 7 is concerned, note that $\bar{X}_n > 0$ holds if and only if its standardized version satisfies

$$W_n^1 = \sqrt{n}\bar{X}_n/\sigma > 0$$

Likewise, $Z_n^q > 0$ if and only if

$$W_n^2 = \sqrt{n}Z_n^q/\sigma_z > 0$$

Therefore,

$$\begin{aligned}\Pr\left\{\bar{X}_n > 0 \wedge Z_n^q > 0\right\} &= \Pr\left\{W_n^1 > 0 \wedge W_n^2 > 0\right\} \\ &\to \Pr\left\{W_\infty^1 > 0 \wedge W_\infty^2 > 0\right\} \\ &= \frac{1}{4} + \frac{1}{2\pi}\arcsin(\rho)\end{aligned}$$

where the last line follows from the Sheppard's Theorem. The probability that these random variables are both negative is the same.

A.5 Proof of Proposition 6

Consider the two intervals separately. Firstly, suppose that $x \in [\beta, 1]$. We have $R(x) = x$ if and only if $x = 1/3$. Notice that $\beta < 1/3$, so this x is in the interval and indeed it is a locally optimal cutoff level. Knowing the values of parameters α and β we can verify that $q = F(1/3)$ is indeed a locally efficient supermajority.

To be globally optimal, one has to make sure that no $x \in [0, \beta]$ is better than $x = 1/3$. Note that

$$\omega(x) \geq \omega(0) = \frac{1}{2}(\alpha + \beta) = 2\frac{1-q}{2-q} = 2\beta \geq 2x$$

where the first inequality follows from the fact that ω (defined in equation A.5) is an increasing function in this interval. The conclusion, $\omega(x) \geq 2x$, implies that $\rho(x)$ is non-decreasing on $[0, \beta]$ and hence no point there is better than $x = \beta$, which itself is strictly worse than $x = 1/3$.

A.6 Proof of Proposition 7

Step 1. Consider another nonnegative random variable Q, that has a c.d.f. $F_Q(\cdot)$ and the mean residual life function $R_Q(\cdot)$. If Q is smaller than X in mean residual life order, $R_Q(x) \leq R(x)$ for all $x \geq 0$, then the variance of Q is finite. To see this note that:

Inequality $R_Q(x) \leq R(x)$ holds if and only if for every $x \geq 0$

$$\frac{\int_x^\infty (1 - F(u))\, du}{\int_x^\infty (1 - F_Q(u))\, du}$$

is a non-decreasing function of x over the set $\left\{x : \int_x^\infty (1 - F_Q(u))\, du > 0\right\}$. Since

$$\frac{\int_x^\infty (1 - F(u))\, du}{\int_x^\infty (1 - F_Q(u))\, du} \geq \frac{\int_0^\infty (1 - F(u))\, du}{\int_0^\infty (1 - F_Q(u))\, du} = \frac{E(X)}{E(Q)} = \zeta > 0$$

it follows that for every $x \geq 0$

$$\int_x^\infty (1 - F(u))\, du \geq \zeta \int_x^\infty (1 - F_Q(u))\, du$$

Take an integral of both sides over all x over the interval $(0, \infty)$ to obtain

$$\int_0^\infty \int_x^\infty (1 - F(u))\, dudx \geq \zeta \int_0^\infty \int_x^\infty (1 - F_Q(u))\, dudx$$

The change of the order of integration implies

$$\int_0^\infty \left(\int_0^u dx\right)(1-F(u))\,du \geq \zeta \int_0^\infty \left(\int_0^u dx\right)(1-F_Q(u))\,du$$

or equivalently

$$\int_0^\infty u(1-F(u))\,du \geq \zeta \int_0^\infty u(1-F_Q(u))\,du$$

Note the integration by parts

$$\int_0^\infty u^2 f(u)\,du = -\lim_{u\to\infty} u^2(1-F(u)) + 2\int_0^\infty u(1-F(u))\,du$$

so that our inequality becomes

$$E\left(X^2\right) + \lim_{u\to\infty} u^2(1-F(u)) \geq \zeta \lim_{u\to\infty} u^2(1-F_Q(u)) + \zeta \int_0^\infty u^2 f_Q(u)\,du$$

The left-hand side is finite because the second moment is finite $E(X^2) < \infty$ and Kolmogorov inequality $u^2(1-F(u)) \leq E(X^2)$. We get that the right-hand side must be finite, so must $E(Q^2)$.

Step 2. Now, for any $\nu > 0$ take a random variable Q with a mean residual life function

$$R_Q(x) = \begin{cases} \lambda & \text{if } x \leq \lambda \\ x & \text{if } x > \lambda \end{cases}$$

The corresponding c.d.f. and p.d.f. are

$$\begin{aligned} F_Q(x) &= \begin{cases} 1-\exp(-x/\lambda) & \text{if } x \leq \lambda \\ 1-\lambda^2\exp(-1)/x^2 & \text{if } x > \lambda \end{cases} \\ f_Q(x) &= \begin{cases} (1/\lambda)\exp(-x/\lambda) & \text{if } x \leq \lambda \\ 2\lambda^2\exp(-1)/x^3 & \text{if } x > \lambda \end{cases} \end{aligned}$$

Note that

$$\begin{aligned} \int_0^\infty u^2 f_Q(u)\,du &= \int_0^\lambda u^2 f_Q(u)\,du + \int_\lambda^\infty u^2 f_Q(u)\,du \\ &= (1/\lambda)\int_0^\lambda u^2\exp(-u/\lambda)\,du + 2\lambda^2\exp(-1)\int_\lambda^\infty (1/u)\,du \\ &= (1/\lambda)\int_0^\lambda u^2\exp(-u/\lambda)\,du + 2\lambda^2\exp(-1)\left(\lim_{u\to\infty}\ln(u) - \ln(\lambda)\right) \\ &= \infty \end{aligned}$$

Step 3. Now suppose that $R(x) > x$ for all x. Select $\lambda = \min_x R(x)$ and note that λ is strictly positive. Note also that $R(x) \geq R_Q(x)$, where $R_Q(x)$ is associated with the random variable Q defined in step 2. By step 2 it has an infinite variance. By step 1 variance of Q must be finite. We obtain a contradiction. Therefore $R(x) \leq x$ for some $x \geq 0$. Since R is a continuous function, $R(x) = x$ for some $x \geq 0$.

References

Aghion, Philippe, Alberto Alesina, Francesco Trebbi, (2004), Endogenous Political Institutions, *The Quarterly Journal of Economics*, Vol. 119, No. 2, pp. 565-611.

Baron, David P. and John A. Ferejohn, (1989), Bargaining in Legislatures, *The American Political Science Review*, Vol. 83, No. 4., pp. 1181-1206.

Barry, Brian, (1965), *Political Argument*, Routledge and Kegan Paul Ltd.

Battaglini, Marco, Rebecca Morton, Thomas Palfrey, (2007), Efficiency, Equity, and Timing of Voting Mechanisms, *American Political Science Review*, 101: 409-424.

Baumgartner, Frederic J., (2003) *Behind Locked Doors: A History of the Papal Elections*. Palgrave Macmillan.

Black, Duncan, (1948), The Decisions of a Committee using a Special Majority, *Econometrica*, Vol. 16, No. 3, pp. 245-261.

Black, Duncan, (1963), *The Theory of Committees and Elections*, Cambridge University Press.

Buchanan, James M. and Gordon Tullock, (1962), *The Calculus of Consent, Logical Foundations of Constitutional Democracy.* Ann Arbor: The University of Michigan Press.

Caplin, Andrew and Barry Nalebuff, (1991), Aggregation and Social Choice: A Mean Voter Theorem, *Econometrica*, Vol. 59, No. 1, pp. 1-23.

Compte Olivier and Philippe Jehiel (2010), Bargaining and Majority Rules: A Collective Search Perspective, *Journal of Political Economy*, Vol. 118, No. 2, pp. 189-221.

David, H. A., (1981), *Order Statistics*, 2nd ed., John Wiley & Sons, Inc.

Ferguson, Thomas S. (1999), Asymptotic joint distribution of sample mean and a sample quantile, mimeo.

Greene, William, (2002), *Econometric analysis.* 5th ed., Pearson Education, Inc.

Holden, Richard T. (2009), Supermajority Voting Rules, mimeo.

May, Kenneth O., (1952), A Set of Independent, Necessary and Sufficient Conditions for Simple Majority Decision, *Econometrica*, 20, 680-4.

Messner, Matthias and Mattias K. Polborn, (2004), Voting on Majority Rules, *Review of Economic Studies*, Vol. 71, No. 1, pp. 115-132

Piketty, Thomas (2000), Voting as Communicating, *Review of Economic Studies*, 67, 169-191.

Ponsati, Clara, and Jozsef Sakovics, (1996), Multiperson Bargaining over Two Alternatives, *Games and Economic Behavior*, 12, 226–244.

Rae, Douglas W. (1969), Decision-Rules and Individual Values in Constitutional Choice, *The American Political Science Review*, Vol. 63, No. 1, pp 40-56.